Contents

1 integration_before_after Theory

Built: 19 July 2018
Parent Theories: normal_rv_hvg

1.1 Definitions

[CDF_def]

```
⊢ ∀p X t. CDF p X t = distribution p X {y | y ≤ t}
```

[cont_CDF_def]

```
⊢ ∀p X. cont_CDF p X ⟺ ∀z. (λx. real (CDF p X x)) contl z
```

[distributed_def]

```
⊢ ∀p M X f.
      distributed p M X f ⟺
      X ∈
      measurable (m_space p,measurable_sets p)
        (m_space M,measurable_sets M) ∧
      f ∈ measurable (m_space M,measurable_sets M) Borel ∧
      AE M {x | 0 ≤ f x} ∧ (distr p M X = density M f)
```

[measurable_CDF_def]

```
⊢ ∀p X.
      measurable_CDF p X ⟺
      (λx. CDF p X x) ∈ measurable borel Borel
```

1.2 Theorems

[ABS_BOUNF_LT]

```
⊢ ∀x k. abs x < k ⟺ -k < x ∧ x < k
```

[after_event_integration]

```
⊢ ∀X Y p M t.
      measure_space M ∧ sigma_finite_measure M ∧ prob_space p ∧
      indep_var p M X M Y ∧
      sigma_finite_measure (distr p M X) ∧
      sigma_finite_measure (distr p M Y) ∧
      (∀t.
           {(w,u) | w < u ∧ 0 ≤ u ∧ u ≤ t} ∈
           measurable_sets (pair_measure M M)) ∧
      (λ(x,y).
           indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
           indicator_fn {w | w < y} x) ∈
      measurable
        (m_space (pair_measure (distr p M X) (distr p M Y)),
         measurable_sets
```

```
        (pair_measure (distr p M X) (distr p M Y))) Borel ∧
  (∀y. {w | w < y} ∈ measurable_sets M) ∧
  (∀n x.
        PREIMAGE X {y | x - 1 / &SUC n < y ∧ y ≤ x} ∩
        p_space p ∈ events p ∧
        PREIMAGE X {y | y ≤ x} ∩ p_space p ∈ events p ∧
        PREIMAGE X {y | y < x} ∩ p_space p ∈ events p ∧
        PREIMAGE X {y | y ≤ x - 1 / &SUC n} ∩ p_space p ∈
        events p) ∧
  (∀x. PREIMAGE X {y | y = x} ∩ p_space p ∈ events p) ∧
  (∀z. (λx. real (CDF p X x)) contl z) ⇒
  (prob p
     (PREIMAGE (λx. (X x,Y x))
        {(w,u) | w < u ∧ 0 ≤ u ∧ u ≤ t} ∩ p_space p) =
   pos_fn_integral (distr p M Y)
     (λy. indicator_fn {u | 0 ≤ u ∧ u ≤ t} y × CDF p X y))
```

[after_set_BIGUNION_IN_MESURABLE_SETS]

```
⊢ ∀t q.
    {u | u < real q} × {w | real q < w ∧ 0 ≤ w ∧ w ≤ t} ∈
    measurable_sets (pair_measure lborel lborel)
```

[after_set_BIGUNION_Q]

```
⊢ ∀t.
    BIGUNION
      {{u | u < real q} × {w | real q < w ∧ 0 ≤ w ∧ w ≤ t} |
       q ∈ Q_set} =
    {(u,a) | u < a ∧ 0 ≤ a ∧ a ≤ t}
```

[after_set_IN_MEASURABLE_SETS_PAIR_lborel]

```
⊢ ∀t.
    {(u,a) | u < a ∧ 0 ≤ a ∧ a ≤ t} ∈
    measurable_sets (pair_measure lborel lborel)
```

[before_event_integration]

```
⊢ ∀X Y p M t.
    measure_space M ∧ sigma_finite_measure M ∧ prob_space p ∧
    indep_var p M X M Y ∧
    sigma_finite_measure (distr p M X) ∧
    sigma_finite_measure (distr p M Y) ∧
    (λ(x,y).
        indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
        indicator_fn {u | x < u} y) ∈
    measurable
      (m_space (pair_measure (distr p M X) (distr p M Y)),
       measurable_sets
         (pair_measure (distr p M X) (distr p M Y))) Borel ∧
    (∀x. {w | w > x} ∈ measurable_sets M) ∧
```

```
(∀x. PREIMAGE Y {y | y ≤ x} ∩ p_space p ∈ events p) ∧
(∀x. PREIMAGE Y {y | y > x} ∩ p_space p ∈ events p) ∧
(∀r.
     {(w,u) | 0 ≤ w ∧ w ≤ r ∧ w < u} ∈
     measurable_sets (pair_measure M M)) ⇒
(prob p
   (PREIMAGE (λx. (X x,Y x))
      {(w,u) | 0 ≤ w ∧ w ≤ t ∧ w < u} ∩ p_space p) =
 pos_fn_integral (distr p M X)
   (λx.
        indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
        (1 - CDF p Y x)))
```

[before_set_BIGUNION_IN_MESURABLE_SETS]

```
⊢ ∀t q.
       {u | u < real q ∧ 0 ≤ u ∧ u ≤ t} × {w | real q < w} ∈
       measurable_sets (pair_measure lborel lborel)
```

[before_set_BIGUNION_Q]

```
⊢ ∀t.
       BIGUNION
         {{u | u < real q ∧ 0 ≤ u ∧ u ≤ t} × {w | real q < w} |
          q ∈ Q_set} =
       {(u,a) | u < a ∧ 0 ≤ u ∧ u ≤ t}
```

[before_set_IN_MEASURABLE_SETS_PAIR_lborel]

```
⊢ ∀t.
       {(u,a) | u < a ∧ 0 ≤ u ∧ u ≤ t} ∈
       measurable_sets (pair_measure lborel lborel)
```

[CDF_after_integration]

```
⊢ ∀X Y p M t.
       measure_space M ∧ sigma_finite_measure M ∧ prob_space p ∧
       indep_var p M X M Y ∧
       sigma_finite_measure (distr p M X) ∧
       sigma_finite_measure (distr p M Y) ∧
       (λ(x,y).
            indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
            indicator_fn {w | w < y} x) ∈
       measurable
         (m_space (pair_measure (distr p M X) (distr p M Y)),
          measurable_sets
            (pair_measure (distr p M X) (distr p M Y))) Borel ∧
       (∀y. {w | w < y} ∈ measurable_sets M) ∧
       (∀n x.
            PREIMAGE X {y | x - 1 / &SUC n < y ∧ y ≤ x} ∩
            p_space p ∈ events p ∧
            PREIMAGE X {y | y ≤ x} ∩ p_space p ∈ events p ∧
```

```
          PREIMAGE X {y | y < x} ∩ p_space p ∈ events p ∧
          PREIMAGE X {y | y ≤ x - 1 / &SUC n} ∩ p_space p ∈
          events p) ∧
     (∀x. PREIMAGE X {y | y = x} ∩ p_space p ∈ events p) ∧
     (∀z. (λx. real (CDF p X x)) contl z) ⇒
     (pos_fn_integral
        (pair_measure (distr p M X) (distr p M Y))
        (λ(x,y).
             indicator_fn {(w,u) | w < u ∧ 0 ≤ u ∧ u ≤ t}
               (x,y)) =
      pos_fn_integral (distr p M Y)
        (λy. indicator_fn {u | 0 ≤ u ∧ u ≤ t} y × CDF p X y))
```

[CDF_after_integration_2]

```
 ⊢ ∀X Y p M fy t.
     measure_space M ∧ sigma_finite_measure M ∧ prob_space p ∧
     indep_var p M X M Y ∧
     sigma_finite_measure (distr p M X) ∧
     sigma_finite_measure (distr p M Y) ∧
     (λ(x,y).
          indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
          indicator_fn {w | w < y} x) ∈
     measurable
       (m_space (pair_measure (distr p M X) (distr p M Y)),
        measurable_sets
          (pair_measure (distr p M X) (distr p M Y))) Borel ∧
     (∀y. {w | w < y} ∈ measurable_sets M) ∧
     distributed p M Y fy ∧
     (∀n x.
          PREIMAGE X {y | x - 1 / &SUC n < y ∧ y ≤ x} ∩
          p_space p ∈ events p ∧
          PREIMAGE X {y | y ≤ x} ∩ p_space p ∈ events p ∧
          PREIMAGE X {y | y < x} ∩ p_space p ∈ events p ∧
          PREIMAGE X {y | y ≤ x - 1 / &SUC n} ∩ p_space p ∈
          events p) ∧
     (∀x. PREIMAGE X {y | y = x} ∩ p_space p ∈ events p) ∧
     (∀z. (λx. real (CDF p X x)) contl z) ⇒
     (pos_fn_integral
        (pair_measure (distr p M X) (distr p M Y))
        (λ(x,y).
             indicator_fn {(w,u) | w < u ∧ 0 ≤ u ∧ u ≤ t}
               (x,y)) =
      pos_fn_integral (density M fy)
        (λy. indicator_fn {u | 0 ≤ u ∧ u ≤ t} y × CDF p X y))
```

[CDF_after_PDF_integration]

```
 ⊢ ∀X Y p M fy t.
     measure_space M ∧ sigma_finite_measure M ∧ prob_space p ∧
     indep_var p M X M Y ∧
```

```
        sigma_finite_measure (distr p M X) ∧
        sigma_finite_measure (distr p M Y) ∧
        (λ (x,y).
             indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
             indicator_fn {w | w < y} x) ∈
        measurable
          (m_space (pair_measure (distr p M X) (distr p M Y)),
           measurable_sets
             (pair_measure (distr p M X) (distr p M Y))) Borel ∧
        (∀ y. {w | w < y} ∈ measurable_sets M) ∧
        distributed p M Y fy ∧
        (λ y. indicator_fn {u | 0 ≤ u ∧ u ≤ t} y × CDF p X y) ∈
        measurable (m_space M,measurable_sets M) Borel ∧
        (∀ y. 0 ≤ CDF p X y) ∧ (∀ y. 0 ≤ fy y) ∧
        (∀ n x.
             PREIMAGE X {y | x - 1 / &SUC n < y ∧ y ≤ x} ∩
             p_space p ∈ events p ∧
             PREIMAGE X {y | y ≤ x} ∩ p_space p ∈ events p ∧
             PREIMAGE X {y | y < x} ∩ p_space p ∈ events p ∧
             PREIMAGE X {y | y ≤ x - 1 / &SUC n} ∩ p_space p ∈
             events p) ∧
        (∀ x. PREIMAGE X {y | y = x} ∩ p_space p ∈ events p) ∧
        (∀ z. (λ x. real (CDF p X x)) contl z) ⇒
        (pos_fn_integral
           (pair_measure (distr p M X) (distr p M Y))
           (λ (x,y).
                indicator_fn {(w,u) | w < u ∧ 0 ≤ u ∧ u ≤ t}
                  (x,y)) =
         pos_fn_integral M
           (λ y.
                fy y ×
                (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y × CDF p X y)))
```

[CDF_after_PDF_integration_general]

```
 ⊢ ∀ X Y p M fy t.
        measure_space M ∧ sigma_finite_measure M ∧ prob_space p ∧
        indep_var p M X M Y ∧
        sigma_finite_measure (distr p M X) ∧
        sigma_finite_measure (distr p M Y) ∧
        (∀ t.
             {(w,u) | w < u ∧ 0 ≤ u ∧ u ≤ t} ∈
             measurable_sets (pair_measure M M)) ∧
        (λ (x,y).
             indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
             indicator_fn {w | w < y} x) ∈
        measurable
          (m_space (pair_measure (distr p M X) (distr p M Y)),
           measurable_sets
             (pair_measure (distr p M X) (distr p M Y))) Borel ∧
```

```
    (∀y. {w | w < y} ∈ measurable_sets M) ∧
    distributed p M Y fy ∧
    (λy. indicator_fn {u | 0 ≤ u ∧ u ≤ t} y × CDF p X y) ∈
    measurable (m_space M,measurable_sets M) Borel ∧
    (∀y. 0 ≤ CDF p X y) ∧ (∀y. 0 ≤ fy y) ∧
    (∀n x.
         PREIMAGE X {y | x - 1 / &SUC n < y ∧ y ≤ x} ∩
         p_space p ∈ events p ∧
         PREIMAGE X {y | y ≤ x} ∩ p_space p ∈ events p ∧
         PREIMAGE X {y | y < x} ∩ p_space p ∈ events p ∧
         PREIMAGE X {y | y ≤ x - 1 / &SUC n} ∩ p_space p ∈
         events p) ∧
    (∀x. PREIMAGE X {y | y = x} ∩ p_space p ∈ events p) ∧
    (∀z. (λx. real (CDF p X x)) contl z) ⇒
    (prob p
       (PREIMAGE (λx. (X x,Y x))
          {(w,u) | w < u ∧ 0 ≤ u ∧ u ≤ t} ∩ p_space p) =
     pos_fn_integral M
       (λy.
            fy y ×
            (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y × CDF p X y)))
```

[CDF_before_integration]

```
 ⊢ ∀X Y p M t.
    measure_space M ∧ sigma_finite_measure M ∧ prob_space p ∧
    indep_var p M X M Y ∧
    sigma_finite_measure (distr p M X) ∧
    sigma_finite_measure (distr p M Y) ∧
    (λ(x,y).
         indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
         indicator_fn {u | x < u} y) ∈
    measurable
      (m_space (pair_measure (distr p M X) (distr p M Y)),
       measurable_sets
         (pair_measure (distr p M X) (distr p M Y))) Borel ∧
    (∀x. {w | w > x} ∈ measurable_sets M) ∧
    (∀x. PREIMAGE Y {y | y ≤ x} ∩ p_space p ∈ events p) ∧
    (∀x. PREIMAGE Y {y | y > x} ∩ p_space p ∈ events p) ⇒
    (pos_fn_integral
       (pair_measure (distr p M X) (distr p M Y))
       (λ(x,y).
            indicator_fn {(w,u) | 0 ≤ w ∧ w ≤ t ∧ w < u}
              (x,y)) =
     pos_fn_integral (distr p M X)
       (λx.
            indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
            (1 - CDF p Y x)))
```

[CDF_before_integration_2]

```
⊢ ∀X Y p M fx t.
      measure_space M ∧ sigma_finite_measure M ∧ prob_space p ∧
      indep_var p M X M Y ∧
      sigma_finite_measure (distr p M X) ∧
      sigma_finite_measure (distr p M Y) ∧
      (λ(x,y).
           indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
           indicator_fn {u | x < u} y) ∈
      measurable
        (m_space (pair_measure (distr p M X) (distr p M Y)),
         measurable_sets
           (pair_measure (distr p M X) (distr p M Y))) Borel ∧
      (∀x. {w | w > x} ∈ measurable_sets M) ∧
      distributed p M X fx ∧
      (∀x. PREIMAGE Y {y | y ≤ x} ∩ p_space p ∈ events p) ∧
      (∀x. PREIMAGE Y {y | y > x} ∩ p_space p ∈ events p) ⇒
      (pos_fn_integral
         (pair_measure (distr p M X) (distr p M Y))
         (λ(x,y).
              indicator_fn {(w,u) | 0 ≤ w ∧ w ≤ t ∧ w < u}
                (x,y)) =
       pos_fn_integral (density M fx)
         (λx.
              indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
              (1 - CDF p Y x)))
```

[CDF_before_PDF_integration]

```
⊢ ∀X Y p M fx t.
      measure_space M ∧ sigma_finite_measure M ∧ prob_space p ∧
      indep_var p M X M Y ∧
      sigma_finite_measure (distr p M X) ∧
      sigma_finite_measure (distr p M Y) ∧
      (λ(x,y).
           indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
           indicator_fn {u | x < u} y) ∈
      measurable
        (m_space (pair_measure (distr p M X) (distr p M Y)),
         measurable_sets
           (pair_measure (distr p M X) (distr p M Y))) Borel ∧
      (∀x. {w | w > x} ∈ measurable_sets M) ∧
      distributed p M X fx ∧
      (∀x. PREIMAGE Y {y | y ≤ x} ∩ p_space p ∈ events p) ∧
      (∀x. PREIMAGE Y {y | y > x} ∩ p_space p ∈ events p) ∧
      (λx. indicator_fn {w | 0 ≤ w ∧ w ≤ t} x × (1 - CDF p Y x)) ∈
      measurable (m_space M,measurable_sets M) Borel ∧
      (∀x. 0 ≤ 1 - CDF p Y x) ∧ (∀x. 0 ≤ fx x) ⇒
      (pos_fn_integral
         (pair_measure (distr p M X) (distr p M Y))
         (λ(x,y).
```

```
          indicator_fn {(w,u) | 0 ≤ w ∧ w ≤ t ∧ w < u}
            (x,y)) =
    pos_fn_integral M
      (λx.
          fx x ×
          (indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
           (1 - CDF p Y x))))
```

[CDF_before_PDF_integration_general]

```
⊢ ∀X Y p M fx t.
      measure_space M ∧ sigma_finite_measure M ∧ prob_space p ∧
      indep_var p M X M Y ∧
      sigma_finite_measure (distr p M X) ∧
      sigma_finite_measure (distr p M Y) ∧
      (λ(x,y).
           indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
           indicator_fn {u | x < u} y) ∈
      measurable
        (m_space (pair_measure (distr p M X) (distr p M Y)),
         measurable_sets
           (pair_measure (distr p M X) (distr p M Y))) Borel ∧
      (∀x. {w | w > x} ∈ measurable_sets M) ∧
      distributed p M X fx ∧
      (∀x. PREIMAGE Y {y | y ≤ x} ∩ p_space p ∈ events p) ∧
      (∀x. PREIMAGE Y {y | y > x} ∩ p_space p ∈ events p) ∧
      (λx. indicator_fn {w | 0 ≤ w ∧ w ≤ t} x × (1 - CDF p Y x)) ∈
      measurable (m_space M,measurable_sets M) Borel ∧
      (∀x. 0 ≤ 1 - CDF p Y x) ∧ (∀x. 0 ≤ fx x) ∧
      {(w,u) | 0 ≤ w ∧ w ≤ t ∧ w < u} ∈
      measurable_sets (pair_measure M M) ⇒
      (prob p
         (PREIMAGE (λx. (X x,Y x))
            {(w,u) | 0 ≤ w ∧ w ≤ t ∧ w < u} ∩ p_space p) =
       pos_fn_integral M
         (λx.
              fx x ×
              (indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
               (1 - CDF p Y x))))
```

[CDF_def2]

```
⊢ ∀p X x.
      prob_space p ∧
      (∀n.
           PREIMAGE X {y | x - 1 / &SUC n < y ∧ y ≤ x} ∩
           p_space p ∈ events p ∧
           PREIMAGE X {y | y ≤ x} ∩ p_space p ∈ events p ∧
           PREIMAGE X {y | y < x} ∩ p_space p ∈ events p ∧
           PREIMAGE X {y | y ≤ x - 1 / &SUC n} ∩ p_space p ∈
           events p) ∧
```

```
PREIMAGE X {y | y = x} ∩ p_space p ∈ events p ∧
(∀z. (λx. real (CDF p X x)) contl z) ⇒
(CDF p X x = prob p (PREIMAGE X {y | y < x} ∩ p_space p))
```

[CDF_integral_indicator]

```
⊢ ∀X p t M.
    random_variable X p (m_space M,measurable_sets M) ∧
    measure_space M ∧ {y | y ≤ t} ∈ measurable_sets M ⇒
    (CDF p X t =
     integral (distr p M X) (indicator_fn {y | y ≤ t}))
```

[CDF_INTERVAL]

```
⊢ ∀p X a b.
    a ≤ b ∧ prob_space p ∧
    PREIMAGE X {y | y ≤ a} ∩ p_space p ∈ events p ∧
    PREIMAGE X {y | a < y ∧ y ≤ b} ∩ p_space p ∈ events p ⇒
    (distribution p X {y | a < y ∧ y ≤ b} =
     CDF p X b - CDF p X a)
```

[CDF_pos]

```
⊢ ∀p X t M.
    random_variable X p (m_space M,measurable_sets M) ∧
    {y | y ≤ t} ∈ measurable_sets M ⇒
    0 ≤ CDF p X t
```

[CDF_pos_fn_integral_indicator]

```
⊢ ∀X p t M.
    random_variable X p (m_space M,measurable_sets M) ∧
    measure_space M ∧ {y | y ≤ t} ∈ measurable_sets M ⇒
    (CDF p X t =
     pos_fn_integral (distr p M X)
       (λx. indicator_fn {y | y ≤ t} x))
```

[CDF_pos_fn_integral_indicator1]

```
⊢ ∀X p t M.
    random_variable X p (m_space M,measurable_sets M) ∧
    measure_space M ∧ (∀t. {y | y ≤ t} ∈ measurable_sets M) ⇒
    (CDF p X t =
     pos_fn_integral (distr p M X)
       (λx. indicator_fn {y | y ≤ t} x))
```

[CONT_PROB_POINT_0]

```
⊢ ∀p X x.
    prob_space p ∧
    (∀n.
         PREIMAGE X {y | x - 1 / &SUC n < y ∧ y ≤ x} ∩
         p_space p ∈ events p ∧
         PREIMAGE X {y | y ≤ x} ∩ p_space p ∈ events p ∧
```

```
          PREIMAGE X {y | y ≤ x - 1 / &SUC n} ∩ p_space p ∈
          events p) ∧
       PREIMAGE X {y | y = x} ∩ p_space p ∈ events p ∧
       (∀z. (λx. real (CDF p X x)) contl z) ⇒
       (prob p (PREIMAGE X {y | y = x} ∩ p_space p) = 0)
```

[CONT_PROB_POINT_EQ_0]

```
 ⊢ ∀p X x.
       prob_space p ∧
       (∀n.
          PREIMAGE X {y | x - 1 / &SUC n < y ∧ y ≤ x} ∩
          p_space p ∈ events p ∧
          PREIMAGE X {y | y ≤ x} ∩ p_space p ∈ events p ∧
          PREIMAGE X {y | y ≤ x - 1 / &SUC n} ∩ p_space p ∈
          events p) ∧
       PREIMAGE X {y | y = x} ∩ p_space p ∈ events p ∧
       (∀z. (λx. real (CDF p X x)) contl z) ⇒
       (real (prob p (PREIMAGE X {y | y = x} ∩ p_space p)) = 0)
```

[CONT_PROB_ZERO_POINT]

```
 ⊢ ∀p X x.
       prob_space p ∧
       (∀n.
          PREIMAGE X {y | x - 1 / &SUC n < y ∧ y ≤ x} ∩
          p_space p ∈ events p ∧
          PREIMAGE X {y | y ≤ x} ∩ p_space p ∈ events p ∧
          PREIMAGE X {y | y ≤ x - 1 / &SUC n} ∩ p_space p ∈
          events p) ∧
       PREIMAGE X {y | y = x} ∩ p_space p ∈ events p ∧
       (∀z. (λx. real (CDF p X x)) contl z) ⇒
       real (prob p (PREIMAGE X {y | y = x} ∩ p_space p)) ≤ 0
```

[distribution_lebesgue_thm2_2_rv]

```
 ⊢ ∀X Y p M' A.
       measure_space M' ∧ sigma_finite_measure M' ∧
       random_variable X p (m_space M',measurable_sets M') ∧
       random_variable Y p (m_space M',measurable_sets M') ∧
       A ∈ measurable_sets (pair_measure M' M') ⇒
       (distribution p (λx. (X x,Y x)) A =
        integral (distr p (pair_measure M' M') (λx. (X x,Y x)))
          (indicator_fn A))
```

[distribution_lebesgue_thm2_distr]

```
 ⊢ ∀X p M A.
       measure_space M ∧
       random_variable X p (m_space M,measurable_sets M) ∧
       A ∈ measurable_sets M ⇒
       (distribution p X A =
        integral (distr p M X) (indicator_fn A))
```

[distribution_lebesgue_thm2_function_2]

```
⊢ ∀X p M′ A.
      random_variable X p
        (m_space (pair_measure M′ M′),
         measurable_sets (pair_measure M′ M′)) ∧
      A ∈ measurable_sets (pair_measure M′ M′) ⇒
      (distribution p X A =
       integral (distr p (pair_measure M′ M′) X)
         (indicator_fn A))
```

[distribution_lebesgue_thm2_pos_fn]

```
⊢ ∀X p M A.
      measure_space M ∧
      random_variable X p (m_space M,measurable_sets M) ∧
      A ∈ measurable_sets M ⇒
      (distribution p X A =
       pos_fn_integral (distr p M X) (indicator_fn A))
```

[eq_sub_ladd]

```
⊢ ∀x y z.
      z ≠ NegInf ∧ z ≠ PosInf ⇒ ((x = y - z) ⇔ (x + z = y))
```

[event_after]

```
⊢ ∀X Y t.
      {w | X w < Y w ∧ Y w < t} =
      PREIMAGE (λx. (X x,Y x)) {(u,a) | u < a ∧ a < t}
```

[event_after1]

```
⊢ ∀X Y t.
      {w | X w < Y w ∧ Y w ≤ t} =
      PREIMAGE (λx. (X x,Y x)) {(u,a) | u < a ∧ a ≤ t}
```

[event_BEFORE]

```
⊢ ∀X Y t.
      {w | X w ≤ t ∧ 0 ≤ X w ∧ X w < Y w} =
      PREIMAGE (λx. (X x,Y x)) {(u,a) | u ≤ t ∧ 0 ≤ u ∧ u < a}
```

[extreal_le_eq]

```
⊢ ∀x y. Normal x ≤ Normal y ⇔ x ≤ y
```

[extreal_real_eq]

```
⊢ ∀x y.
      x ≠ PosInf ∧ x ≠ NegInf ∧ y ≠ PosInf ∧ y ≠ NegInf ⇒
      ((real x = real y) ⇔ (x = y))
```

[extreal_real_le]

$\vdash \forall x\ y.$
$x \neq$ PosInf $\wedge\ x \neq$ NegInf $\wedge\ y \neq$ PosInf $\wedge\ y \neq$ NegInf $\Rightarrow$
(real $y \leq$ real $x \iff y \leq x$)

[extreal_real_lt]

$\vdash \forall x\ y.$
$x \neq$ PosInf $\wedge\ x \neq$ NegInf $\wedge\ y \neq$ PosInf $\wedge\ y \neq$ NegInf $\Rightarrow$
(real $y <$ real $x \iff y < x$)

[extreal_real_sub_eq]

$\vdash \forall x\ y\ z.$
$x \neq$ PosInf $\wedge\ x \neq$ NegInf $\wedge\ y \neq$ PosInf $\wedge\ y \neq$ NegInf $\wedge$
$z \neq$ PosInf $\wedge\ z \neq$ NegInf $\Rightarrow$
((real z = real x - real y) $\iff$ (z = x - y))

[EXTREAL_SUM_IMAGE_FUN_MUL]

$\vdash \forall s.$
FINITE $s \Rightarrow$
$\forall f\ Y.$
($\forall x.\ x \in s \Rightarrow f\ x \neq$ NegInf) $\vee$
($\forall x.\ x \in s \Rightarrow f\ x \neq$ PosInf) $\Rightarrow$
$\forall y.$
SIGMA ($\lambda x.$ Normal ($Y\ y$) $\times\ f\ x$) s =
Normal ($Y\ y$) $\times$ SIGMA $f\ s$

[IN_REST]

$\vdash \forall x\ s.\ x \in$ REST $s \iff x \in s \wedge x \neq$ CHOICE s

[indicator_fn_not_eq_infty]

$\vdash \forall s\ x.$ indicator_fn $s\ x \neq$ PosInf $\wedge$ indicator_fn $s\ x \neq$ NegInf

[indicator_mul_pos_le]

$\vdash \forall A\ B\ x\ y.\ 0 \leq$ indicator_fn $A\ x \times$ indicator_fn $B\ y$

[indicator_of_indicator_after]

$\vdash \forall x\ y\ t.$
indicator_fn $\{(w,u) \mid w < u \wedge 0 \leq u \wedge u \leq t\}$ (x,y) =
indicator_fn $\{u \mid 0 \leq u \wedge u \leq t\}$ y $\times$
indicator_fn $\{w \mid w < y\}$ x

[indicator_of_indicator_before]

$\vdash \forall x\ y\ t.$
indicator_fn $\{(u,w) \mid u \leq t \wedge 0 \leq u \wedge u < w\}$ (x,y) =
indicator_fn $\{u \mid 0 \leq u \wedge u \leq t\}$ x $\times$
indicator_fn $\{w \mid x < w\}$ y

[integral_pos_fn_distr]

```
⊢ ∀M p X A.
      measure_space M ∧
      random_variable X p (m_space M,measurable_sets M) ∧
      A ∈ measurable_sets M ⇒
      (pos_fn_integral (distr p M X) (indicator_fn A) =
       integral (distr p M X) (indicator_fn A))
```

[integral_pos_fn_indicator]

```
⊢ ∀m A.
      measure_space m ⇒
      (integral m (indicator_fn A) =
       pos_fn_integral m (indicator_fn A))
```

[LE_UNION_GT]

```
⊢ ∀p X t.
      prob_space p ⇒
      (PREIMAGE X 𝓤(:real) ∩ p_space p =
       PREIMAGE X {y | y ≤ t} ∩ p_space p ∪
       PREIMAGE X {y | y > t} ∩ p_space p)
```

[lemma1]

```
⊢ ∀M Y B f.
      measure_space M ∧ (∀x. x ∈ m_space M ⇒ 0 ≤ f x) ⇒
      ((λy.
            pos_fn_integral M
              (λx. Normal (real (indicator_fn B y)) × f x)) =
       (λy.
            Normal (real (indicator_fn B y)) ×
            pos_fn_integral M (λx. f x)))
```

[lemma2]

```
⊢ ∀M Y B f.
      measure_space M ∧ (∀x. x ∈ m_space M ⇒ 0 ≤ f x) ⇒
      ((λy.
            pos_fn_integral M
              (λx. Normal (real (indicator_fn B y)) × f x)) =
       (λy. indicator_fn B y × pos_fn_integral M (λx. f x)))
```

[lemma2_indicator_mul_after]

```
⊢ ∀X Y p M t.
      measure_space M ∧ sigma_finite_measure M ∧ prob_space p ∧
      indep_var p M X M Y ∧
      sigma_finite_measure (distr p M X) ∧
      sigma_finite_measure (distr p M Y) ∧
      (λ(x,y).
           indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
```

```
      indicator_fn {w | w < y} x) ∈
   measurable
     (m_space (pair_measure (distr p M X) (distr p M Y)),
      measurable_sets
        (pair_measure (distr p M X) (distr p M Y))) Borel ⇒
   (pos_fn_integral
      (pair_measure (distr p M X) (distr p M Y))
      (λ (x,y).
           indicator_fn {(w,u) | w < u ∧ 0 ≤ u ∧ u ≤ t}
             (x,y)) =
    pos_fn_integral (distr p M Y)
      (λ y.
           indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
           pos_fn_integral (distr p M X)
             (λ x. indicator_fn {w | w < y} x)))
```

[lemma2_indicator_mul_before]

```
⊢ ∀ X Y p M t.
     measure_space M ∧ sigma_finite_measure M ∧ prob_space p ∧
     indep_var p M X M Y ∧
     sigma_finite_measure (distr p M X) ∧
     sigma_finite_measure (distr p M Y) ∧
     (λ (x,y).
          indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
          indicator_fn {u | x < u} y) ∈
     measurable
       (m_space (pair_measure (distr p M X) (distr p M Y)),
        measurable_sets
          (pair_measure (distr p M X) (distr p M Y))) Borel ⇒
     (pos_fn_integral
        (pair_measure (distr p M X) (distr p M Y))
        (λ (x,y).
             indicator_fn {(w,u) | 0 ≤ w ∧ w ≤ t ∧ w < u}
               (x,y)) =
      pos_fn_integral (distr p M X)
        (λ x.
             indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
             pos_fn_integral (distr p M Y)
               (λ y. indicator_fn {u | x < u} y)))
```

[lemma3]

```
⊢ ∀ M B f.
     measure_space M ∧ (∀ x. x ∈ m_space M ⇒ 0 ≤ f x) ⇒
     ((λ y. pos_fn_integral M (λ x. indicator_fn B y × f x)) =
      (λ y. indicator_fn B y × pos_fn_integral M (λ x. f x)))
```

[lemma3_indicator_mul_after]

```
⊢ ∀ X Y p t.
     prob_space p ∧ indep_var p lborel X lborel Y ⇒
```

```
(pos_fn_integral
   (pair_measure (distr p lborel X) (distr p lborel Y))
   (λ(x,y).
        indicator_fn {(w,u) | w < u ∧ 0 ≤ u ∧ u ≤ t}
          (x,y)) =
 pos_fn_integral (distr p lborel Y)
   (λy.
        indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
        pos_fn_integral (distr p lborel X)
          (λx. indicator_fn {w | w < y} x)))
```

[lemma3_indicator_mul_before]

```
⊢ ∀X Y p t.
      prob_space p ∧ indep_var p lborel X lborel Y ⇒
      (pos_fn_integral
         (pair_measure (distr p lborel X) (distr p lborel Y))
         (λ(x,y).
              indicator_fn {(w,u) | 0 ≤ w ∧ w ≤ t ∧ w < u}
                (x,y)) =
       pos_fn_integral (distr p lborel X)
         (λx.
              indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
              pos_fn_integral (distr p lborel Y)
                (λy. indicator_fn {u | x < u} y)))
```

[lemma_indicator_mul_after]

```
⊢ ∀X Y p M t.
      measure_space M ∧ sigma_finite_measure M ∧ prob_space p ∧
      indep_var p M X M Y ∧
      sigma_finite_measure (distr p M X) ∧
      sigma_finite_measure (distr p M Y) ∧
      (λ(x,y).
           indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
           indicator_fn {w | w < y} x) ∈
      measurable
        (m_space (pair_measure (distr p M X) (distr p M Y)),
         measurable_sets
           (pair_measure (distr p M X) (distr p M Y))) Borel ⇒
      (pos_fn_integral
         (pair_measure (distr p M X) (distr p M Y))
         (λ(x,y).
              indicator_fn {(w,u) | w < u ∧ 0 ≤ u ∧ u ≤ t}
                (x,y)) =
       pos_fn_integral (distr p M Y)
         (λy.
              pos_fn_integral (distr p M X)
                (λx.
                     indicator_fn {u | 0 ≤ u ∧ u ≤ t} y ×
                     indicator_fn {w | w < y} x)))
```

[lemma_indicator_mul_before]

```
⊢ ∀X Y p M t.
      measure_space M ∧ sigma_finite_measure M ∧ prob_space p ∧
      indep_var p M X M Y ∧
      sigma_finite_measure (distr p M X) ∧
      sigma_finite_measure (distr p M Y) ∧
      (λ(x,y).
           indicator_fn {u | 0 ≤ u ∧ u ≤ t} x ×
           indicator_fn {w | x < w} y) ∈
      measurable
        (m_space (pair_measure (distr p M X) (distr p M Y)),
         measurable_sets
           (pair_measure (distr p M X) (distr p M Y))) Borel ⇒
      (pos_fn_integral
         (pair_measure (distr p M X) (distr p M Y))
         (λ(x,y).
              indicator_fn {(w,u) | 0 ≤ w ∧ w ≤ t ∧ w < u}
                (x,y)) =
       pos_fn_integral (distr p M X)
         (λx.
              pos_fn_integral (distr p M Y)
                (λy.
                     indicator_fn {u | 0 ≤ u ∧ u ≤ t} x ×
                     indicator_fn {w | x < w} y)))
```

[MEASURE_SPACE_BIGUNION_Q]

```
⊢ ∀m s.
      measure_space m ∧
      (∀n. n ∈ Q_set ⇒ s n ∈ measurable_sets m) ⇒
      BIGUNION (IMAGE s Q_set) ∈ measurable_sets m
```

[minus_x_not_infty]

```
⊢ ∀x. x ≠ PosInf ∧ x ≠ NegInf ⇒ -x ≠ NegInf ∧ -x ≠ PosInf
```

[mul_extreal_not_infty]

```
⊢ ∀x y.
      x ≠ PosInf ∧ x ≠ NegInf ∧ y ≠ PosInf ∧ y ≠ NegInf ⇒
      x × y ≠ PosInf ∧ x × y ≠ NegInf
```

[pos_fn_integral_density]

```
⊢ ∀f g M.
      measure_space M ∧
      f ∈ measurable (m_space M,measurable_sets M) Borel ∧
      AE M {x | 0 ≤ f x} ∧
      g ∈ measurable (m_space M,measurable_sets M) Borel ∧
      (∀x. 0 ≤ g x) ⇒
      (pos_fn_integral (density M f) (λx. max 0 (g x)) =
       pos_fn_integral M (λx. max 0 (f x × g x)))
```

[pos_fn_integral_density_1]

```
⊢ ∀f g M.
      measure_space M ∧
      f ∈ measurable (m_space M,measurable_sets M) Borel ∧
      (∀x. 0 ≤ f x) ∧ AE M {x | 0 ≤ f x} ∧
      g ∈ measurable (m_space M,measurable_sets M) Borel ∧
      (∀x. 0 ≤ g x) ⇒
      (pos_fn_integral (density M f) (λx. g x) =
       pos_fn_integral M (λx. f x × g x))
```

[pos_fn_integral_distr'_x]

```
⊢ ∀t f M M′.
      measure_space M ∧ measure_space M′ ∧
      t ∈
      measurable (m_space M,measurable_sets M)
        (m_space M′,measurable_sets M′) ∧
      (λx. x) ∈
      measurable
        (m_space (distr M M′ t),measurable_sets (distr M M′ t))
        Borel ∧ (∀x. 0 ≤ x) ⇒
      (pos_fn_integral (distr M M′ t) (λx. max 0 x) =
       pos_fn_integral M (λx. max 0 (t x)))
```

[pos_fn_integral_distr'_x2]

```
⊢ ∀t f M M′.
      prob_space M ∧ measure_space M′ ∧
      t ∈
      measurable (p_space M,events M)
        (m_space M′,measurable_sets M′) ∧
      (λx. x) ∈
      measurable
        (m_space (distr M M′ t),measurable_sets (distr M M′ t))
        Borel ∧ (∀x. 0 ≤ x) ⇒
      (pos_fn_integral (distr M M′ t) (λx. max 0 x) =
       pos_fn_integral M (λx. max 0 (t x)))
```

[pos_fn_integral_distr'x_random_variable]

```
⊢ ∀t f M.
      prob_space M ∧
      t ∈
      measurable (p_space M,events M)
        (m_space lborel,measurable_sets lborel) ∧
      f ∈
      measurable
        (m_space (distr M lborel t),
         measurable_sets (distr M lborel t)) Borel ∧
      (∀x. 0 ≤ f x) ⇒
      (pos_fn_integral (distr M lborel t) (λx. max 0 (f x)) =
       pos_fn_integral M (λx. max 0 (f (t x))))
```

[pos_fn_integral_fun_mul]

```
⊢ ∀M Y f.
      measure_space M ∧ (∀x. x ∈ m_space M ⇒ 0 ≤ f x) ∧
      (∀y. 0 ≤ Y y) ⇒
      ((λy. pos_fn_integral M (λx. Normal (Y y) × f x)) =
       (λy. Normal (Y y) × pos_fn_integral M (λx. f x)))
```

[pos_fn_integral_gt_CDF]

```
⊢ ∀M p X t.
      random_variable X p (m_space M,measurable_sets M) ∧
      PREIMAGE X {y | y ≤ t} ∩ p_space p ∈ events p ∧
      PREIMAGE X {y | y > t} ∩ p_space p ∈ events p ∧
      measure_space M ∧ (∀t. {y | y > t} ∈ measurable_sets M) ⇒
      (pos_fn_integral (distr p M X)
         (λx. indicator_fn {y | y > t} x) =
       1 - CDF p X t)
```

[pos_simple_fn_integral_fun_mul]

```
⊢ ∀m f s a x Y y.
      measure_space m ∧ pos_simple_fn m f s a x ∧ (∀y. 0 ≤ Y y) ⇒
      pos_simple_fn m (λx. Normal (Y y) × f x) s a
        (λi. Y y × x i) ∧
      (pos_simple_fn_integral m s a (λi. Y y × x i) =
       Normal (Y y) × pos_simple_fn_integral m s a x)
```

[prob_after]

```
⊢ ∀X Y p fy t.
      prob_space p ∧ indep_var p lborel X lborel Y ∧
      distributed p lborel Y fy ∧ (∀y. 0 ≤ fy y) ∧
      cont_CDF p X ∧ measurable_CDF p X ⇒
      (prob p
         (PREIMAGE (λx. (X x,Y x))
            {(w,u) | w < u ∧ 0 ≤ u ∧ u ≤ t} ∩ p_space p) =
       pos_fn_integral lborel
         (λy.
              fy y ×
              (indicator_fn {u | 0 ≤ u ∧ u ≤ t} y × CDF p X y)))
```

[PROB_AT_POINT1]

```
⊢ ∀p X x n.
      prob_space p ∧
      PREIMAGE X {y | x - 1 / &SUC n < y ∧ y ≤ x} ∩ p_space p ∈
      events p ∧ PREIMAGE X {y | y = x} ∩ p_space p ∈ events p ⇒
      prob p (PREIMAGE X {y | y = x} ∩ p_space p) ≤
      distribution p X {y | x - 1 / &SUC n < y ∧ y ≤ x}
```

[prob_before]

```
⊢ ∀X Y p fx t.
      prob_space p ∧ indep_var p lborel X lborel Y ∧
      distributed p lborel X fx ∧ (∀x. 0 ≤ fx x) ∧
      measurable_CDF p Y ⇒
      (prob p
         (PREIMAGE (λx. (X x,Y x))
            {(w,u) | 0 ≤ w ∧ w ≤ t ∧ w < u} ∩ p_space p) =
       pos_fn_integral lborel
         (λx.
              fx x ×
              (indicator_fn {w | 0 ≤ w ∧ w ≤ t} x ×
               (1 - CDF p Y x))))
```

[prob_event]

```
⊢ ∀X p A.
      random_variable X p
        (m_space lborel,measurable_sets lborel) ∧
      A ∈ measurable_sets lborel ⇒
      (prob p (PREIMAGE X A ∩ p_space p) =
       integral (distr p lborel X) (indicator_fn A))
```

[prob_event_2indep_rv]

```
⊢ ∀X Y p M′ A.
      measure_space M′ ∧ sigma_finite_measure M′ ∧
      prob_space p ∧ indep_var p M′ X M′ Y ∧
      A ∈ measurable_sets (pair_measure M′ M′) ⇒
      (prob p (PREIMAGE (λx. (X x,Y x)) A ∩ p_space p) =
       integral (distr p (pair_measure M′ M′) (λx. (X x,Y x)))
         (indicator_fn A))
```

[prob_event_2indep_rv_1]

```
⊢ ∀X Y p M′ A.
      measure_space M′ ∧ sigma_finite_measure M′ ∧
      prob_space p ∧ indep_var p M′ X M′ Y ∧
      A ∈ measurable_sets (pair_measure M′ M′) ∧
      sigma_finite_measure (distr p M′ X) ∧
      sigma_finite_measure (distr p M′ Y) ⇒
      (prob p (PREIMAGE (λx. (X x,Y x)) A ∩ p_space p) =
       integral (pair_measure (distr p M′ X) (distr p M′ Y))
         (indicator_fn A))
```

[prob_event_2indep_rv_PREIMAGE_A]

```
⊢ ∀X Y p M′ A B.
      measure_space M′ ∧ sigma_finite_measure M′ ∧
      prob_space p ∧ indep_var p M′ X M′ Y ∧
      A ∈ measurable_sets (pair_measure M′ M′) ∧
      sigma_finite_measure (distr p M′ X) ∧
```

```
      sigma_finite_measure (distr p M' Y) ⇒
      (prob p (PREIMAGE (λx. (X x,Y x)) A ∩ p_space p) =
       pos_fn_integral
         (pair_measure (distr p M' X) (distr p M' Y))
         (indicator_fn A))
```

[prob_event_2rv]

```
 ⊢ ∀X Y p M' A.
      measure_space M' ∧ sigma_finite_measure M' ∧
      random_variable X p (m_space M',measurable_sets M') ∧
      random_variable Y p (m_space M',measurable_sets M') ∧
      A ∈ measurable_sets (pair_measure M' M') ⇒
      (prob p (PREIMAGE (λx. (X x,Y x)) A ∩ p_space p) =
       integral (distr p (pair_measure M' M') (λx. (X x,Y x)))
         (indicator_fn A))
```

[prob_event_gen]

```
 ⊢ ∀X M p A.
      random_variable X p (m_space M,measurable_sets M) ∧
      A ∈ measurable_sets M ∧ measure_space M ⇒
      (prob p (PREIMAGE X A ∩ p_space p) =
       integral (distr p M X) (indicator_fn A))
```

[prob_event_le]

```
 ⊢ ∀X p t M.
      random_variable X p (m_space M,measurable_sets M) ∧
      measure_space M ∧ {y | y ≤ t} ∈ measurable_sets M ⇒
      (prob p (PREIMAGE X {y | y ≤ t} ∩ p_space p) =
       integral (distr p M X) (indicator_fn {y | y ≤ t}))
```

[prob_gt_1_le]

```
 ⊢ ∀p X t.
      prob_space p ∧
      PREIMAGE X {y | y ≤ t} ∩ p_space p ∈ events p ∧
      PREIMAGE X {y | y > t} ∩ p_space p ∈ events p ⇒
      (prob p (PREIMAGE X {y | y > t} ∩ p_space p) =
       1 - prob p (PREIMAGE X {y | y ≤ t} ∩ p_space p))
```

[prob_gt_CDF]

```
 ⊢ ∀p X t.
      prob_space p ∧
      PREIMAGE X {y | y ≤ t} ∩ p_space p ∈ events p ∧
      PREIMAGE X {y | y > t} ∩ p_space p ∈ events p ⇒
      (prob p (PREIMAGE X {y | y > t} ∩ p_space p) =
       1 - CDF p X t)
```

[prob_gt_pos_fn_integral]

```
⊢ ∀p X t M.
      random_variable X p (m_space M,measurable_sets M) ∧
      measure_space M ∧ (∀t. {y | y > t} ∈ measurable_sets M) ⇒
      (prob p (PREIMAGE X {y | y > t} ∩ p_space p) =
       pos_fn_integral (distr p M X)
         (λx. indicator_fn {y | y > t} x))
```

[PROB_le_eq]

```
⊢ ∀p X x.
      prob_space p ∧
      (∀n.
           PREIMAGE X {y | x - 1 / &SUC n < y ∧ y ≤ x} ∩
           p_space p ∈ events p ∧
           PREIMAGE X {y | y ≤ x} ∩ p_space p ∈ events p ∧
           PREIMAGE X {y | y < x} ∩ p_space p ∈ events p ∧
           PREIMAGE X {y | y ≤ x - 1 / &SUC n} ∩ p_space p ∈
           events p) ∧
      PREIMAGE X {y | y = x} ∩ p_space p ∈ events p ∧
      (∀z. (λx. real (CDF p X x)) contl z) ⇒
      (prob p (PREIMAGE X {y | y ≤ x} ∩ p_space p) =
       prob p (PREIMAGE X {y | y < x} ∩ p_space p))
```

[prob_neq_infty]

```
⊢ ∀p s.
      prob_space p ∧ s ∈ events p ⇒
      prob p s ≠ PosInf ∧ prob p s ≠ NegInf
```

[psfis_fun_mul]

```
⊢ ∀m f a Y y.
      measure_space m ∧ a ∈ psfis m f ∧ (∀y. 0 ≤ Y y) ⇒
      Normal (Y y) × a ∈ psfis m (λx. Normal (Y y) × f x)
```

[REAL_LE_NEG1_0]

```
⊢ -1 ≤ 0
```

[REAL_LE_NEG1_1]

```
⊢ -1 ≤ 1
```

[REAL_LT_NEG1_0]

```
⊢ -1 < 0
```

[REAL_LT_NEG1_1]

```
⊢ -1 < 1
```

[REAL_MAX_COMM]

```
⊢ ∀A B. max A B = max B A
```

[REAL_MIN_COMM]

```
⊢ ∀A B. min A B = min B A
```

[real_of_0]

```
⊢ real 0 = 0
```

[set_le_gt_disjoint]

```
⊢ ∀t. DISJOINT {y | y > t} {y | y ≤ t}
```

[SIGMA_ALGEBRA_FN_Q]

```
⊢ ∀a.
      sigma_algebra a ⇔
      subset_class (space a) (subsets a) ∧ {} ∈ subsets a ∧
      (∀s. s ∈ subsets a ⇒ space a DIFF s ∈ subsets a) ∧
      ∀f.
          f ∈ (Q_set -> subsets a) ⇒
          BIGUNION (IMAGE f Q_set) ∈ subsets a
```

[sigma_algebra_pair_distr_lborel]

```
⊢ ∀X Y p.
      sigma_algebra
        (m_space
           (pair_measure (distr p lborel X) (distr p lborel Y)),
         measurable_sets
           (pair_measure (distr p lborel X) (distr p lborel Y)))
```

[sigma_algebra_pair_lborel]

```
⊢ sigma_algebra
     (m_space (pair_measure lborel lborel),
      measurable_sets (pair_measure lborel lborel))
```

[sigma_finite_measure_distr]

```
⊢ ∀p X.
      prob_space p ∧
      random_variable X p
        (m_space lborel,measurable_sets lborel) ⇒
      sigma_finite_measure (distr p lborel X)
```